

The number of fiberings of a surface bundle over a surface

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Abstract

For a closed manifold M , let $\text{Fib}(M)$ be the number of distinct fiberings of M as a fiber bundle with fiber a closed surface. In this paper we give the first computation of $\text{Fib}(M)$ where $1 < \text{Fib}(M) < \infty$ but M is not a product. In particular, we prove $\text{Fib}(M) = 2$ for the Atiyah-Kodaira manifold and any finite cover of a trivial surface bundle. We also give an example where $\text{Fib}(M) = 4$.

1 Introduction

Let M be a closed manifold. We will call the following number the *fibering number* of M

$$\text{Fib}(M) = \# \left\{ \begin{array}{l} S_g \rightarrow M \rightarrow B \text{ a surface bundle :} \\ S_g \text{ is some genus } g > 1 \text{ surface} \\ \text{and } B \text{ a closed manifold} \end{array} \right\} / \sim \quad (1)$$

where two such bundles are equivalent if and only if there is fiber-bundle isomorphism between them.

In the case $\dim(M) = 3$, Thurston [Thu86] classified all possible bundle structures of the same total space using the Thurston norm. The result states that $\text{Fib}(M) = \infty$ if and only if $\dim H^1(M; \mathbb{Q}) > 1$. The \mathbb{Q} -points in the ‘fibered cone’ of $H^1(M; \mathbb{R})$ are in one-to-one correspondence with distinct fiberings.

In the case $\dim(M) = 4$ and $\chi(M) > 0$, Johnson [Joh99] proved that $\text{Fib}(M) < \infty$. We can also deduce an upper bound for this number depending only on $\chi(M)$. For any $N > 1$, Salter [Sal15a] constructed an example with $\text{Fib}(M) > N$. He [Sal15b] also gave some conditions on the monodromy of a given fibering of M so that $\text{Fib}(M) = 1$. For example, in the case of a nontrivial bundle where the monodromy is in the Johnson kernel, he proved that $\text{Fib}(M) = 1$.

One beautiful example of a multi-fibered 4-manifold is the Atiyah-Kodaira manifold M_{AK} , see [Ati15] and [Hir15]. Atiyah’s construction of M_{AK} (see Section 3 below for details) has at least two different fiberings.

$$S_6 \rightarrow M_{AK} \rightarrow S_{129}$$

and

$$S_{321} \rightarrow M_{AK} \rightarrow S_3.$$

A natural question comes out: are these the only two fiberings?

Theorem 1.1 (Fibering number of M_{AK}). $\text{Fib}(M_{AK}) = 2$.

Another case where we can compute the fibering number is the case of a finite cover of a product $B \times F$.

Theorem 1.2 (Finite cover of a trivial bundle). *Let E be regular finite cover of a trivial bundle $B \times F$ where $g(B) > 1$ and $g(F) > 1$, then $\text{Fib}(E) = 1$*

Salter [Sal15a] gave the first construction of a surface bundle over surface where $\text{Fib}(M_S) \geq 4$, see Section 5 for details. In this paper we also compute $\text{Fib}(M_S)$.

Theorem 1.3 (Salter's 4 fibering example). $\text{Fib}(M_S) = 4$.

However, all the examples Salter constructed have $\text{Fib}(M)$ a power of 2. Therefore, we ask the following question.

Question 1.4 (3 fiberings construction). *Is there a surface bundle over surface M so that $\text{Fib}(M) = 3$?*

Organization of the paper

In Section 2, we will give descriptions of M_{AK} , a geometric description and the monodromy representation. Then in Section 3 we will prove our main theorem: $\text{Fib}(M_{AK}) = 2$. Section 4 we prove that when M is a finite cover of a trivial bundle, $\text{Fib}(M) = 2$. In section 5 we show that $\text{Fib}(M_S) = 4$ for M_S as constructed in Salter [Sal15b].

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2 Description of M_{AK} and the uniqueness problem

In this section, we will describe the Atiyah-Kodaira manifold M_{AK} and also its monodromy representation. In the end, we will post a question of whether different AK examples are diffeomorphic.

2.1 Geometric construction of M_{AK}

We follow the construction in [Mor01].

Let S_3 be the genus 3 surface, and let τ be a free $\mathbb{Z}/2\mathbb{Z}$ action on S_3 . The trivial bundle $S_3 \times S_3$ has 2 sections: Γ_d the graph of identity and Γ_τ the graph of τ . Since the action is free, the two sections are disjoint. The surjective homomorphism $\pi_1(S_3) \rightarrow H_1(S_3; \mathbb{Z}/2)$ gives us a cover $S_{129} \xrightarrow{i} S_3$. We have the following exact sequence.

$$1 \rightarrow \pi_1(S_{129}) \xrightarrow{i_*} \pi_1(S_3) \rightarrow H_1(S_3; \mathbb{Z}/2) \rightarrow 1$$

The pull-back surface bundle $i^*(S_3 \times S_3) \cong S_{129} \times S_3$ also has 2 sections $S_d = i^*(\Gamma_d)$ and $S_\tau = i^*(\Gamma_\tau)$. We have $[S_d] = \text{graph}(i)$ and $[S_\tau] = \text{graph}(\tau \circ i)$.

$$H^2(S_{129} \times S_3; \mathbb{Z}/2) \cong H^2(S_{129}; \mathbb{Z}/2) \oplus [H^1(S_{129}; \mathbb{Z}/2) \otimes H^1(S_3; \mathbb{Z}/2)] \oplus H^2(S_3; \mathbb{Z}/2)$$

By the cover we choose, we have $[S_d] = [S_{129}]$ and $[S_\tau] = [S_{129}] \in H^2(S_{129} \times S_3; \mathbb{Z}/2)$. Therefore

$$[S_d] + [S_\tau] = 0 \in H_2(S_{129} \times S_3; \mathbb{Z}/2).$$

Let us denote $PD([S_d] + [S_\tau])$ as the Poincare dual of $[S_d] + [S_\tau]$. By Poincare duality, we have that

$$PD([S_d] + [S_\tau]) = 0 \in H^2(S_{129} \times S_3; \mathbb{Z}/2).$$

Let $M = S_{129} \times S_3 - S_d - S_\tau$. We have the following long exact sequence of the cohomology of the relative pair $(S_{129} \times S_3, M)$:

$$H^1(S_{129} \times S_3, M; \mathbb{Z}/2) \rightarrow H^1(S_{129} \times S_3; \mathbb{Z}/2) \rightarrow H^1(M; \mathbb{Z}/2) \xrightarrow{\phi} H^2(S_{129} \times S_3, M; \mathbb{Z}/2) \xrightarrow{T} H^2(S_{129} \times S_3; \mathbb{Z}/2)$$

By the Thom isomorphism theorem, we have

$$H^1(S_{129} \times S_3, M; \mathbb{Z}/2) = 0$$

and

$$H^2(S_{129} \times S_3, M; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Now $T(1, 0) = PD[S_d]$ and $T(0, 1) = PD[S_\tau]$. Therefore

$$T(1, 1) = 0 \in H^2(S_{129} \times S_3; \mathbb{Z}/2)$$

So $\phi^{-1}(1, 1)$ is not empty in $H^1(M; \mathbb{Z}/2)$. By the following isomorphism, we have that $H^1(M; \mathbb{Z}/2)$ classifies $\mathbb{Z}/2$ covers of M .

$$\text{Hom}(\pi(M), \mathbb{Z}/2) \cong H^1(M; \mathbb{Z}/2)$$

Therefore $\phi^{-1}(1, 1)$ classifies the $\mathbb{Z}/2$ branched covers of $S_{129} \times S_3$ branched over $[S_d] + [S_\tau]$. Let M_{AK} be one of them.

These branched covers are parametrized by $H^1(S_{129} \times S_3; \mathbb{Z}/2)$. But how does $H^1(S_{129} \times S_3; \mathbb{Z}/2)$ change the monodromy? We will answer this question in the remark of the next subsection.

We also pose a question about Atiyah-Kodaira construction as the following:

Question 2.1 (Uniqueness of AK example). *Are the different choices of branched covers occurring in the construction of Atiyah-Kodaira manifold diffeomorphic?*

2.2 Monodromy description

Let $\text{PMod}_{g,n}$ be the pure mapping class group of $S_{g,n}$, i.e. the isotopy classes of S_g that fix n points individually. Let $\text{Mod}_{g,n}$ be the mapping class group of $S_{g,n}$, i.e. the isotopy classes of S_g that fix n points as a set. We have a *generalized Birman exact sequence* as the following, see e.g. [FM12].

$$1 \rightarrow \pi_1(\text{PConf}(S_g)) \rightarrow \text{PMod}_{g,n} \rightarrow \text{Mod}_g \rightarrow 1$$

The two disjoint sections of the bundle $S_3 \times S_3$ give us a map $(id, \tau) : S_3 \rightarrow \text{PConf}_2(S_3)$, therefore we have a monodromy representation:

$$\pi_1(S_3) \rightarrow \pi_1(\text{PConf}_2(S_3)) \rightarrow \text{PMod}_{3,2}.$$

Let $b \in S_3$ and $b' = \tau(b)$. The $\mathbb{Z}/2$ branched covers of S_3 branched over b and b' are parametrized by $H^1(S_3; \mathbb{Z}/2)$. Pick any branched cover $S_{6,2} \rightarrow S_{3,2}$ with deck transformation σ . Let $\text{PMod}_{6,2}^\sigma$ be the stabilizer of σ in $\text{PMod}_{6,2}$. We will have a map as the following.

$$\text{PMod}_{6,2}^\sigma \rightarrow \text{Mod}_{3,2}$$

Since the kernel $\pi_1(S_{129})$ acts trivially on $H^1(S_3; \mathbb{Z}/2)$, we will have that the monodromy $\pi_1(S_{129}) \rightarrow \pi_1(S_3) \rightarrow \text{Mod}_{3,2}$ can lift to $\text{PMod}_{6,2}^\sigma$.

Problem 2.2. *The lift of the monodromy is not unique! Let $\{a_i, b_i\}$ be the generators of $\pi_1(S_{129})$, and let $\rho : \pi_1(S_{129}) \rightarrow \text{Mod}_6$ be a monodromy representation. Because σ is commutative with any element in $\{A_i = \rho(a_i), B_i = \rho(b_i)\}$, we could multiply σ to a subset of $\{A_i, B_i\}$ to get a new monodromy representation. For example, $\{A_i\sigma, B_i\}$ is a new monodromy representation. Among the different monodromies, are the total spaces of all the bundles diffeomorphic to each other?*

3 The proof of Theorem 1.1

In this section, we will give a proof of Theorem 1.1 by describing the monodromy action on homology and computing $H^1(M_{AK}; \mathbb{Q})$.

3.1 Lift of a square of point pushing

Let a be the loop in Figure 1. We have that $\text{Push}(a) = T_x T_y^{-1}$.

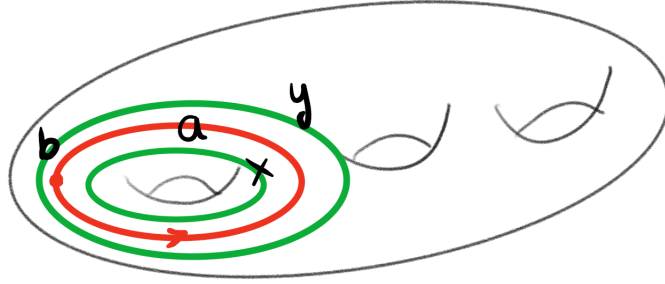


Figure 1: Point pushing

Because the $\mathbb{Z}/2$ cover is branched over the point b , one of the curves x or y will lift to two copies and the other will lift to a single curve. Also a will have two lifts, which we call \bar{a} and \tilde{a} . Let $\text{Lift}(\text{Push}(a)^2)$ be the lift of point-pushing action on S_6 .

Lemma 3.1. *For $c \in H_1(S_6)$, the action $\text{Lift}(\text{Push}(a)^2)$ on c has the following 2 possibilities:*

$$\text{Lift}(\text{Push}(a)^2)(c) = c \pm i(c, \tilde{a} - \bar{a})(\tilde{a} - \bar{a})$$

or

$$\text{Lift}(\text{Push}(a)^2)(c) = \sigma_*(c \pm i(c, \tilde{a} - \bar{a})(\tilde{a} - \bar{a}))$$

Proof. Suppose $\text{Lift}(x) = \bar{x} \cup \tilde{x}$ and $\text{Lift}(y) = y'$. By looking at the action locally, we have that $\text{Lift}(T_x^2) = T_{\bar{x}}^2 T_{\tilde{x}}^2$ and $\text{Lift}(T_y) = T_{y'}$. Therefore

$$\text{Lift}(\text{Push}(a)^2) = \text{Lift}(T_x T_y^{-1}) = T_{\bar{x}}^2 T_{\tilde{x}}^2 T_{y'}^{-1}.$$

We know that as a homology class $y' = \bar{x} + \tilde{x}$, we have the following computation.

$$\begin{aligned} T_{\bar{x}}^2 T_{\tilde{x}}^2 T_{y'}^{-1}(c) &= c - i(c, y')y' + i(c, \bar{x})2\bar{x} + i(c, \tilde{x})2\tilde{x} \\ &= c + i(c, \bar{x})(\bar{x} - \tilde{x}) + i(c, \tilde{x})(\tilde{x} - \bar{x}) \\ &= c + i(c, \tilde{x} - \bar{x})(\tilde{x} - \bar{x}) \end{aligned} \tag{2}$$

It is not hard to see that the two lifts of a and x are homotopic, therefore we have

$$\text{Lift}(\text{Push}(a)^2)(c) = c + i(c, \tilde{a} - \bar{a})(\tilde{a} - \bar{a})$$

The case where $\text{Lift}(y) = \bar{y} \cup \tilde{y}$ and $\text{Lift}(x) = x'$, we have

$$\text{Lift}(\text{Push}(a)^2)(c) = c - i(c, \tilde{a} - \bar{a})(\tilde{a} - \bar{a})$$

□

3.2 $H^1(S_3; \mathbb{Q}) \subset H^1(S_6; \mathbb{Q})$

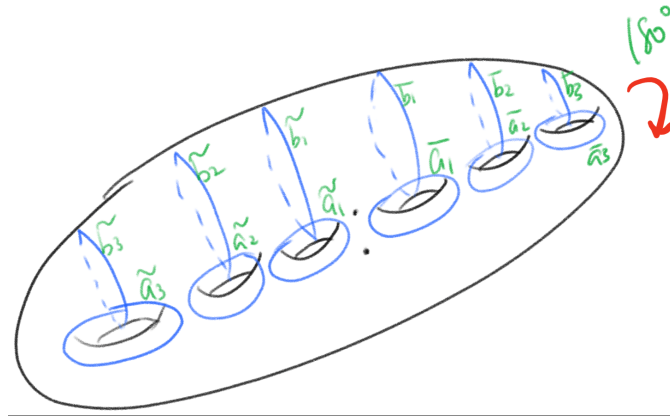


Figure 2: The action σ

With the action σ on S_6 , we could decompose $H_1(S_6; \mathbb{Q})$ by eigenvalues of σ_* . The deck transformation σ is an involution, therefore the eigenvalue of σ_* is $\{\pm 1\}$. Let H^+ be the eigenspace of σ associated with eigenvalue $+1$ and H^- the eigenspace of σ associated with eigenvalue -1 .

$$H_1(S_6; \mathbb{Q}) = H^- \oplus H^+.$$

Let $S_6 \xrightarrow{p} S_3$ be the branched cover on one fiber. Since $H^1 = \text{Hom}(H_1, \mathbb{Q})$, a cohomology class is the same as a functional on H_1 .

Claim 3.2. *A functional $f : H_1(S_6; \mathbb{Q}) \rightarrow \mathbb{Q}$ belongs to $p^*H^1(S_3; \mathbb{Q})$ if and only if $H^- \subset \ker(f)$.*

Proof. By the transfer map, we know that $H^+ \cong H_1(S_3; \mathbb{Q})$. The map $H_1(S_6) \rightarrow H_1(S_3)$ is the same as the projection to H^+ coordinate. □

With the above Figure 2, we have a geometric description of a basis $\{\bar{a}_1, \tilde{a}_1, \dots, \}$ of $H_1(S_6)$.

3.3 The π_{129} -invariant cohomology

Pick a free $\mathbb{Z}/2$ action τ on S_3 as in the Figure 3. Let ϕ be the monodromy representation of a_1^2 in Mod_6 . We have that $\phi(a_1^2) = \text{Lift}(\text{Push}(a_1)^2 \text{Push}(\tau(a_1))^2)$.

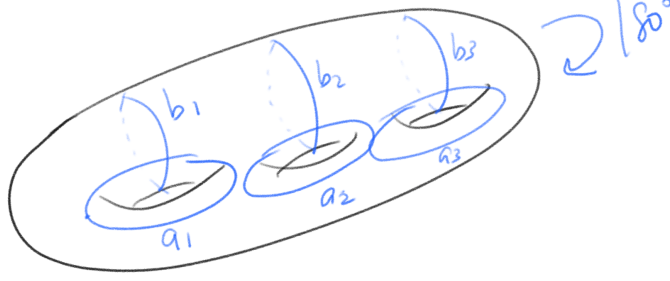


Figure 3: The action τ

Lemma 3.3. *The invariant cohomology $f \in H^1(S_6)$ under the action of a_1^2 has to satisfy $f(\tilde{a}_1 - \bar{a}_1) = 0$ and $f(\tilde{a}_3 - \bar{a}_3) = 0$*

Proof. By Claim 3.1

$$\phi(a_1^2)(c) = c \pm i(b, \tilde{a}_1 - \bar{a}_1)(\tilde{a}_1 - \bar{a}_1) \pm i(b, \tilde{a}_3 - \bar{a}_3)(\tilde{a}_3 - \bar{a}_3)$$

Case 1: For any b in $H_1(S_6)$,

$$f(b) = f(b) \pm i(b, \tilde{a}_1 - \bar{a}_1)f(\tilde{a}_1 - \bar{a}_1) \pm i(b, \tilde{a}_3 - \bar{a}_3)f(\tilde{a}_3 - \bar{a}_3).$$

Equivalently,

$$i(b, \tilde{a}_1 - \bar{a}_1)f(\tilde{a}_1 - \bar{a}_1) \pm i(b, \tilde{a}_3 - \bar{a}_3)f(\tilde{a}_3 - \bar{a}_3) = 0$$

However, $\tilde{a}_1 - \bar{a}_1$ and $\tilde{a}_3 - \bar{a}_3$ are independent elements in $H_1(S_6)$, so we can find b such that $i(b, \tilde{a}_1 - \bar{a}_1) = 0$ and $i(b, \tilde{a}_3 - \bar{a}_3) = 1$. Therefore we must have $f(\tilde{a}_3 - \bar{a}_3) = 0$ and $f(\tilde{a}_1 - \bar{a}_1) = 0$

Case 2: For any b in $H_1(S_6)$,

$$f(b) = f(\sigma_*(b)) \pm i(b, \tilde{a}_1 - \bar{a}_1)f(\sigma_*(\tilde{a}_1 - \bar{a}_1)) \pm i(b, \tilde{a}_3 - \bar{a}_3)f(\sigma_*(\tilde{a}_3 - \bar{a}_3)).$$

If we set $b = \tilde{a}_1$, $b = \tilde{a}_3$, we get

$$f(\tilde{a}_1) = f(\bar{a}_1)$$

$$f(\tilde{a}_3) = f(\bar{a}_3)$$

□

Lemma 3.4. *We have the following isomorphism.*

$$H^1(S_6; \mathbb{Q})^{\pi_1(S_{129})} \cong H^1(S_3; \mathbb{Q})$$

Proof. We use the same argument as Lemma 3.3 on $(b_2 a_1)^2$, we get that

$$f(\widetilde{b_2 + a_1 - b_2 + a_1}) = 0.$$

Since we already have $f(\tilde{a}_1 - \bar{a}_1) = 0$, we get that $f(\tilde{b}_2 - \bar{b}_2) = 0$.

From the above discussion, we have that $\dim(H^1(S_6; \mathbb{Q})^{\pi_1(S_{129})}) \leq 7$. We also have that $p^*H^1(S_3; \mathbb{Q}) \subset H^1(S_6; \mathbb{Q})^{\pi_1(S_{129})}$, therefore $\dim(H^1(S_6; \mathbb{Q})^{\pi_1(S_{129})}) \geq 6$. However, we have an exact sequence

$$1 \rightarrow H^1(S_6; \mathbb{Q})^{\pi_1(S_{129})} \rightarrow H^1(M_{AK}; \mathbb{Q}) \rightarrow H^1(S_{129}; \mathbb{Q}) \rightarrow 1$$

which means

$$\dim(H^1(M_{AK}; \mathbb{Q})) = \dim(H^1(S_{129}; \mathbb{Q})) + \dim(H^1(S_6; \mathbb{Q})^{\pi_1(S_{129})})$$

Since M_{AK} is a Kahler manifold, it has even first betti number. So we have $\dim(H^1(S_6; \mathbb{Q})^{\pi_1(S_{129})}) \geq 6$ which means

$$H^1(S_6; \mathbb{Q})^{\pi_1(S_{129})} \cong H^1(S_3; \mathbb{Q})$$

□

3.4 Proof of the Lemma ‘A condition on two fiberings’

Lemma 3.5. *Given any total space M of a surface bundle over a surface, if there are two different coverings $M \xrightarrow{p_1} B_1$ and $M \xrightarrow{p_2} B_2$, then $p_1^*(H^1(B_1; \mathbb{Q})) \cap p_2^*(H^1(B_2; \mathbb{Q})) = \{0\}$.*

Proof. This is Lemma 3.3 in Salter [Sal15a].

□

Lemma 3.6 (A condition on two fiberings). *Let $S_{h_1} \rightarrow M \xrightarrow{p_1} S_{g_1}$ be a surface bundle over surface where $h_1, g_1 > 1$. Let $S_{h_2} \rightarrow M \xrightarrow{p_2} S_{g_2}$ be another bundle structure with $g_2, h_2 > 1$. Let $(p_1, p_2) : M \rightarrow S_{g_1} \times S_{g_2}$. If*

$$(p_1, p_2)^*H^1(S_{g_1} \times S_{g_2}; \mathbb{Q}) \cong H^1(M; \mathbb{Q})$$

and if

$$(p_1, p_2)^*H^2(S_{g_1} \times S_{g_2}; \mathbb{Q}) \rightarrow H^2(M; \mathbb{Q})$$

is injective, then $\text{Fib}(M) = 2$.

Proof. Suppose there exists a third fibering $F \rightarrow M \xrightarrow{p} B$. By Lemma 3.5, for every nonzero element $x \in H^1(B; \mathbb{Q})$ there exists $a \neq 0 \in p_1^*H^1(S_{g_1}; \mathbb{Q})$ and $b \neq 0 \in p_2^*H^1(S_{g_2}; \mathbb{Q})$ such that

$$p^*(x) = a + b \in H^1(M; \mathbb{Q}) \cong p_1^*H^1(S_{g_1}; \mathbb{Q}) \oplus p_2^*H^1(S_{g_2}; \mathbb{Q})$$

Since $\chi(M) > 0$ and $\chi(F) < 0$, we have $\chi(B) < 0$ implying $g(B) > 1$. Therefore we have another element $y \neq 0 \in H^1(B; \mathbb{Q})$ not a multiple of x but satisfying that

$$x \cup y = 0 \in H^2(B; \mathbb{Q}).$$

Suppose that

$$p^*(y) = c + d \in H^1(M; \mathbb{Q}) \cong p_1^*H^1(S_{g_1}; \mathbb{Q}) \oplus p_2^*H^1(S_{g_2}; \mathbb{Q})$$

We have $x \cup y = 0$ implying that

$$(a + b)(c + d) = 0 \in (p_1, p_2)^*H^2(S_{g_1} \times S_{g_2}; \mathbb{Q}) \subset H^2(M; \mathbb{Q})$$

By Kunneth formula,

$$H^2(S_{g_1} \times S_{g_2}; \mathbb{Q}) \cong H^2(S_{g_1}; \mathbb{Q}) \oplus H^1(S_{g_1}; \mathbb{Q}) \otimes H^1(S_{g_2}; \mathbb{Q}) \oplus H^2(S_{g_2}; \mathbb{Q})$$

we will have $ad + bc = 0$ and $ac = bd = 0$. By the property of tensor product, the only possibility is that $c = ka$ and $d = kb$. In this case, y is a multiple of x , which contradicts to our assumption on y . Therefore, the result follows. □

Proof of Theorem 1.1. Since the top cohomology map $H^4(S_3 \times S_{129}; \mathbb{Q}) \rightarrow H^4(M_{AK}; \mathbb{Q})$ is an isomorphism, by Poincare duality, we have that the cohomology with \mathbb{Q} coefficients is injective on every dimension. By Lemma 3.4 we also have

$$H^1(M_{AK}; \mathbb{Q}) = H^1(S_3; \mathbb{Q}) \oplus H^1(S_{129}; \mathbb{Q}).$$

Therefore M_{AK} satisfies the assumption of Lemma 3.6, which shows $\text{Fib}(M_{AK}) = 2$. \square

4 Fiber number of a finite cover of a trivial bundle

In this section, we will prove Theorem 1.2.

Let E be a regular finite cover of $B \times F$, where $g(B) > 1$ and $g(F) > 1$. Let $p_1 : E \rightarrow B$ and let $p_2 : E \rightarrow F$. We will denote $\text{Im}(p_1)$ the image of $p_{1*} : \pi_1(E) \rightarrow \pi_1(B)$ and $\text{Im}(p_2)$ the image of $p_{2*} : \pi_1(E) \rightarrow \pi_1(F)$.

Lemma 4.1.

$$H^1(E; \mathbb{Q}) \cong H^1(\text{Im}(p_1); \mathbb{Q}) \oplus H^1(\text{Im}(p_2); \mathbb{Q})$$

Proof. Let $\pi_1(\tilde{F})$ be the kernel of $\pi_1(E) \rightarrow \text{Im}(p_1)$. We have the following diagram.

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(\tilde{F}) & \longrightarrow & \pi_1(E) & \longrightarrow & \text{Im}(p_1) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow = \\ 1 & \longrightarrow & \text{Im}(p_2) & \longrightarrow & \text{Im}(p_1) \times \text{Im}(p_2) & \longrightarrow & \text{Im}(p_1) \longrightarrow 1 \end{array} \quad (3)$$

We have that $H^1(\tilde{F}; \mathbb{Q})^{\text{Im}(p_1)} = H^1(\tilde{F}; \mathbb{Q})^{\pi_1(E)}$, but we also know that $\pi_1(E) \rightarrow \text{Im}(p_2)$ is surjective. Therefore

$$H^1(\tilde{F}; \mathbb{Q})^{\pi_1(E)} \subset H^1(\tilde{F}; \mathbb{Q})^{\text{Im}(p_2)} \cong H^1(\text{Im}(p_2); \mathbb{Q}).$$

However, since the action of $\pi_1(E)$ on $\text{Im}(p_2)$ is trivial, we know that $H^1(\text{Im}(p_2); \mathbb{Q}) \subset H^1(\tilde{F}; \mathbb{Q})^{\pi_1(E)}$. Therefore we have that $H^1(\tilde{F}; \mathbb{Q})^{\text{Im}(p_1)} \cong H^1(\text{Im}(p_2); \mathbb{Q})$.

Combining the Serre spectral sequence on the top exact sequence of (3), we have the following exact sequence.

$$0 \rightarrow H^1(\text{Im}(p_1); \mathbb{Q}) \rightarrow H^1(E; \mathbb{Q}) \rightarrow H^1(\tilde{F}; \mathbb{Q})^{\text{Im}(p_1)} \cong H^1(\text{Im}(p_2); \mathbb{Q}) \rightarrow 0$$

Therefore our lemma follows. \square

Proof of Theorem 1.2. $\pi_1(E)$ is a finite index subgroup of $\text{Im}(p_1) \times \text{Im}(p_2)$, therefore $H^4(\text{Im}(p_1) \times \text{Im}(p_2); \mathbb{Q}) \rightarrow H^4(E; \mathbb{Q})$ is an isomorphism. By Poincare duality, the map on the cohomology is injective for every dimension. Specifically $H^2(\text{Im}(p_1) \times \text{Im}(p_2); \mathbb{Q}) \subset H^2(E; \mathbb{Q})$. Therefore E satisfies the assumption of Lemma 3.6, which shows $\text{Fib}(E) = 2$. \square

5 An Example with Exactly 4 Fiberings

Now we do another example of Salter [Sal15b] that has exactly 4 different fiberings. This example is a section sum of two trivial bundles with diagonal sections.

Let Δ be the diagonal in $S_g \times S_g$. Let $M_S = (S_g \times S_g - \Delta) \cup_{\phi} (S_g \times S_g - \Delta)$, where ϕ is the identification of the boundary of $S_g \times S_g - \Delta$. Each copy of $S_g \times S_g - \Delta$ has two fiberings p_1 and p_2 where p_i means the projection onto the i th coordinate. Therefore M_S has 4 obvious fiberings:

$$1) (p_1, p_1) \quad 2) (p_1, p_2) \quad 3) (p_2, p_1) \quad 4) (p_2, p_2).$$

Remark 5.1. When you want to write down the map (p_1, p_2) and (p_2, p_1) , you have to perturb the function so that they match on the diagonal. See Section 2 in Salter [Sal15b].

Lemma 5.2.

$$H^1(S_g \times S_g - \Delta; \mathbb{Q}) \cong p_1^*(H^1(S_g; \mathbb{Q})) \oplus p_2^*(H^1(S_g; \mathbb{Q}))$$

Proof. This lemma has been proved in [Che16]. □

Lemma 5.3. There exists the following exact sequence.

$$0 \rightarrow H^1(M_S; \mathbb{Q}) \rightarrow H^1(S_g; \mathbb{Q}) \oplus H^1(S_g; \mathbb{Q}) \oplus H^1(S_g; \mathbb{Q}) \oplus H^1(S_g; \mathbb{Q}) \xrightarrow{add} H^1(S_g; \mathbb{Q}) \rightarrow 0$$

and

$$0 \rightarrow H^2(M_S; \mathbb{Q}) \xrightarrow{i^*} H^2(S_g \times S_g - \Delta; \mathbb{Q}) \oplus H^2(S_g \times S_g - \Delta; \mathbb{Q})$$

Proof. Let E_1 and E_2 be the two copies of $S_g \times S_g - \Delta$ in the construction of M_S . The intersection is called N , which is a circle bundle over S_g . This bundle has Euler number $2 - 2g$, therefore

$$H^1(N; \mathbb{Q}) = H^1(S_g; \mathbb{Q}).$$

The map

$$H_1(N; \mathbb{Q}) = H_1(S_g; \mathbb{Q}) \rightarrow H_1(S_g \times S_g; \mathbb{Q}) = H_1(S_g; \mathbb{Q}) \oplus H_1(S_g; \mathbb{Q})$$

is the diagonal. Therefore

$$H^1(S_g \times S_g; \mathbb{Q}) = H^1(S_g; \mathbb{Q}) \oplus H^1(S_g; \mathbb{Q}) \rightarrow H^1(N; \mathbb{Q}) = H^1(S_g; \mathbb{Q})$$

is the addition of the two elements (dual to the diagonal map).

So we have a long exact sequence coming from the Mayer-Vietoris pair (E_1, E_2) as:

$$\begin{aligned} 0 \longrightarrow H^1(M_S; \mathbb{Q}) \rightarrow H^1(S_g \times S_g - \Delta; \mathbb{Q}) \oplus H^1(S_g \times S_g - \Delta; \mathbb{Q}) \xrightarrow{s^*} H^1(N; \mathbb{Q}) \longrightarrow \\ \longrightarrow H^2(M_S; \mathbb{Q}) \xrightarrow{i^*} H^2(S_g \times S_g - \Delta; \mathbb{Q}) \oplus H^2(S_g \times S_g - \Delta; \mathbb{Q}) \end{aligned}$$

We know s^* is surjective from our discussion, therefore we have that i^* is injective. □

Notation. In what follows, we use x to represent an element in $H^1(S_g; \mathbb{Q})$. In

$$H^1(S_g \times S_g - \Delta; \mathbb{Q}) \oplus H^1(S_g \times S_g - \Delta; \mathbb{Q}) \cong H^1(S_g; \mathbb{Q}) \oplus H^1(S_g; \mathbb{Q}) \oplus H^1(S_g; \mathbb{Q}) \oplus H^1(S_g; \mathbb{Q}),$$

we use the notation x, x', \bar{x}, \bar{x}' to represent the pullbacks of the 4 different projections of x . Notice that in the surjection onto $H^1(N; \mathbb{Q})$, they all map to the same element x itself.

Claim 5.4. *With respect to the 4 fiberings that we have above, the pullback of first cohomology is the span of the following elements.*

- 1) $\{x - \bar{x}\}$ for any $x \in H^1(S_g; \mathbb{Q})$;
- 2) $\{x - \bar{x}'\}$ for any $x \in H^1(S_g; \mathbb{Q})$;
- 3) $\{x' - \bar{x}\}$ for any $x \in H^1(S_g; \mathbb{Q})$;
- 4) $\{x' - \bar{x}'\}$ for any $x \in H^1(S_g; \mathbb{Q})$;

We will need the following algebraic lemma:

Lemma 5.5. *In*

$$\wedge H^1(S_g \times S_g - \Delta; \mathbb{Q}) \xrightarrow{\text{cup product}} H^2(S_g \times S_g - \Delta; \mathbb{Q})$$

for two independent elements $x, y \in H^1(S_g \times S_g - \Delta; \mathbb{Q})$, if $x \cup y = 0$ then for some $i \in \{1, 2\}$, we have $x, y \in p_i^*(H^1(S_g; \mathbb{Q}))$

Proof. This lemma is proved in [Che16]. □

Proof of Theorem 1.3. From the naturality of cup product we have the following commutative diagram.

$$\begin{array}{ccc} \wedge H^1(M_S; \mathbb{Q}) & \longrightarrow & \wedge(H^1(S_g \times S_g - \Delta; \mathbb{Q}) \oplus H^1(S_g \times S_g - \Delta; \mathbb{Q})) \\ \downarrow & & \downarrow \\ H^2(M_S; \mathbb{Q}) & \longrightarrow & H^2(S_g \times S_g - \Delta; \mathbb{Q}) \oplus H^2(S_g \times S_g - \Delta; \mathbb{Q}) \end{array}$$

The element in $H^1(M_S; \mathbb{Q})$ is a combination $x + y' + \bar{z} + \bar{w}'$ such that $x + y + z + w = 0$ in $H^1(M_S)$ and the image in $H^1(S_g \times S_g - \Delta; \mathbb{Q}) \oplus H^1(S_g \times S_g - \Delta; \mathbb{Q})$ is $(x + y', z + w')$. Suppose we have another fibering $S_h \rightarrow E \xrightarrow{p} B$. Since $\chi(M_S) > 0$ and $\chi(S_h) < 0$, by computation $\chi(B) < 0$ implying that $g(B) > 1$. Therefore there exist independent $b, b' \in H^1(B; \mathbb{Q})$ so that $b \cup b' = 0$. Let $p^*(b) = x_1 + y'_1 + \bar{z}_1 + \bar{w}'_1$ and $p^*(b') = x_2 + y'_2 + \bar{z}_2 + \bar{w}'_2$.

By Lemma 5.3, $p^*(b) \cup p^*(b') = 0$ if and only if $(x_1 + y'_1)(x_2 + y'_2) = 0$ and $(z_1 + w'_1)(z_2 + w'_2) = 0$. Using the Lemma 5.5, we get the following clasfication: either of 1) or 1') is true and either of 2) or 2') is true

- 1) $x_1 + y'_1$ and $x_2 + y'_2$ are dependent
- 1') $x_1 = x_2 = 0$ or $y_1 = y_2 = 0$
- 2) $z_1 + w'_1$ and $z_2 + w'_2$ are dependent
- 2') $z_1 = z_2 = 0$ or $w_1 = w_2 = 0$

Claim 5.6. *There must be two elements satisfy 1') and 2') in the subspace H .*

Proof. Suppose there is an element $x + y' + \bar{z} + \bar{w}' \in H$ that has x, y, x and w all nonzero. If $a \in H$ such that $a(x + y' + \bar{z} + \bar{w}') = 0$, then by Lemma 5.5, we have $k, l \in \mathbb{Q}$ so that $a = k(x + y') + l(\bar{z} + \bar{w}')$. However $k(x + y') + l(\bar{z} + \bar{w}')$ only spans a 2-dimensional space, contradicting to the fact that $\dim H > 3$.

Therefore every element $x + y' + \bar{z} + \bar{w}' \in H$ has one coordinate zero. If there is an element $x + y' + \bar{z} + \bar{w}' \in H$ that has two coordinates zero, this element belongs to one of the 4 known fiberings. So we get that every element has exactly one coordinate zero.

If two elements in $a, b \in H$ have different coordinates zero, we could find a linear combination $ka + lb \in H$ that has nonzero coordinate, contradicting to the above argument. Therefore all elements in H have the same coordinate equal to zero.

With loss of generality suppose any element $x + y' + \bar{z} + \bar{w}' \in H$ has $w' = 0$. Suppose there were independent $x_1 + y'_1 + \bar{z}_1, x_2 + y'_2 + \bar{z}_2 \in H$ such that $(x_1 + y'_1 + \bar{z}_1) \cup (x_2 + y'_2 + \bar{z}_2) = 0$. We should have $x_2 + y'_2 = k(x_1 + y'_1)$ by Lemma 5.3. However we should also have $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$. herefore $\bar{z}_2 = k\bar{z}_1$, which means $x_2 + y'_2 + \bar{z}_2 = k(x_1 + y'_1 + \bar{z}_1)$. Which is a contradiction. \square

If we have 1') and 2'), $H = p^*(H^1(B; \mathbb{Q}))$ will intersect with one of 4 known fiberings. By Lemma 3.5 we don't get a new fibering. \square

References

- [Ati15] MF Atiyah. The signature of fibre-bundles. *Global Analysis (Papers in Honor of K. Kodaira)*, pages 73–84, 2015.
- [Che16] Lei Chen. The universal n -pointed surface bundle only has n sections. *arXiv preprint arXiv:1611.04624*, 2016.
- [FM12] B Farb and D Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.
- [Hir15] F Hirzebruch. The signature of ramified coverings. *Global analysis (papers in honor of K. Kodaira)*, pages 253–265, 2015.
- [Joh99] F. E. A. Johnson. A rigidity theorem for group extensions. *Arch. Math. (Basel)*, 73(2):81–89, 1999.
- [Mor01] Shigeyuki Morita. *Geometry of Characteristic Classes (Translations of Mathematical Monographs)*. American Mathematical Society, 5 2001.
- [Sal15a] N Salter. Cup products, the Johnson homomorphism and surface bundles over surfaces with multiple fiberings. *Algebr. Geom. Topol.*, 15(6):3613–3652, 2015.
- [Sal15b] Nick Salter. Surface bundles over surfaces with arbitrarily many fiberings. *Geom. Topol.*, 19:2901–2923, 2015.
- [Thu86] William P Thurston. A norm for the homology of 3-manifolds. *Memoirs of the American Mathematical Society*, 59(339):99–130, 1986.

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